Fuzzy and rough formal concept analysis: a survey

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Formal Concept Analysis (FCA) is a mathematical technique that has been extensively applied to Boolean data in knowledge discovery, information retrieval, web mining, etc. applications. During the past years, the research on extending FCA theory to cope with imprecise and incomplete information made significant progress. In this paper, we give a systematic overview of the more than 120 papers published between 2003 and 2011 on FCA with fuzzy attributes and rough FCA. We applied traditional FCA as a text-mining instrument to 1072 papers mentioning FCA in the abstract. These papers were formatted in pdf files and using a thesaurus with terms referring to research topics, we transformed them into concept lattices. These lattices were used to analyze and explore the most prominent research topics within the FCA with fuzzy attributes and rough FCA research communities. FCA turned out to be an ideal metatechnique for representing large vol umes of unstructured texts.

Keywords: formal concept analysis (FCA); rough sets; fuzzy attributes; knowledge discovery in databases; text mining; exploratory data analysis; information retrieval; systematic literature overview

1. Introduction

Formal Concept Analysis (FCA) was introduced in the early 1980s by Rudolf Wille as a mathematical theory (Wille 1982) taking its roots in the work of Barbut and Monjardet (1970) and Birkhoff (1973). FCA is concerned with the formalization of concepts and conceptual thinking and has been applied in many disciplines such as software engineering, knowledge discovery, and information retrieval during the last 15 years. The mathematical foundation of FCA is described by Ganter and Wille (1999) and introductory courses were written by Wolff (1994) and Wille (1997).

A textual overview of part of the literature published until the year 2004 on the mathematical and philosophical background of FCA, some of the applications of FCA in the information retrieval and knowledge discovery field and in logic and AI is given by Priss (2006). A comparison of algorithms for generating concept lattices was performed by Kuznetsov and Obiedkov (2002). An overview of available FCA software is provided by Tilley (2004). Carpineto and Romano (2004) present an overview of FCA applications in information retrieval. In Tilley and Eklund (2007), an overview of 47 FCA-based software engineering papers is presented. The authors categorized these papers according to the 10 categories as defined in the ISO 12207 software engineering standard and visualized them in a concept lattice. In

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Lakhal and Stumme (2005), a survey on FCA-based association rule mining techniques is given. Poelmans et al. (2010, 2013a, 2013b) give an extensive overview of FCA-based models and their applications in knowledge discovery. Poelmans et al. (2012a) discuss papers published on FCA in the information retrieval field. Doerfel et al. (2012) investigated patterns and communities in co-author and citation contexts obtained from publications in the International Conference on Formal Concept Analysis (ICFCA), International Conference on Conceptual Structures (ICCS) and Concept Lattices and Applications (CLA).

In this paper, we describe how we used FCA to create a visual overview of the existing literature on concept analysis on uncertain data published between the years 2003 and 2011. The core contributions of this paper are as follows. We visually represent the literature on FCA using concept lattices, in which the objects are the scientific papers and the attributes are the relevant terms available in the title, keywords, and abstract of the papers. We developed a toolset with a central FCA component that we use to index the papers with a thesaurus containing terms related to FCA research and to generate the lattices. We give an overview of the literature on extending FCA with fuzzy logic and rough set theory. We zoom in on and give an extensive overview of the papers published between 2003 and 2011 on using FCA with fuzzy attributes and rough FCA in knowledge discovery and data mining (DM), information retrieval, and ontology engineering.

The remainder of this paper is composed as follows. In Section 2, we introduce the essentials of FCA theory and the knowledge browsing environment we developed to support this literature analysis. In Section 3, we describe the dataset used. In Section 4, we visualize the FCA literature on knowledge discovery, information retrieval, ontology engineering and scalability using concept lattices. In Section 5, we give an overview of the literature on FCA with fuzzy attributes. In Section 6, we present an overview of the literature on rough FCA. In Section 7, we give an overview of extensions of FCA with Axiomatic Fuzzy Set (AFS) algebra. Section 8 concludes the paper.

2. FCA

2.1. FCA essentials

FCA (Ganter and Wille 1999; Wille 1982) is a recent mathematical framework that underlies many methods of knowledge discovery and data analysis. The starting point of FCA is a triple of sets (X, Y, I) called a *formal context* where $I \subseteq X \times Y$ is a binary relation. This triple can be considered as a cross table consisting of set of rows X (called objects), columns Y (called attributes) and crosses representing relation I. An example of a cross table is displayed in Table 1 where objects are papers, attributes are terms, and incidence relation shows how terms occur in papers. In what follows, scientific papers (i.e. the objects) are related (i.e. the crosses) to a number of terms (i.e. the attributes); here, a paper is related to a term if the title or abstract of the paper contains this term. The dataset in Table 1 is an excerpt of the one we

Table 1. Example of a formal context.

	Browsing	Mining	Software	Web services	FCA	Information retrieval
Paper 1	×	×	×		×	
Paper 2			×		×	×
Paper 3		×		×	×	
Paper 4	×		×		×	
Paper 5				×	×	×

used in our research. Given a formal context, FCA then derives formal concepts and orders them according to a subconcept/superconcept relation. This order makes a lattice that can be visualized by a line diagram.

The notion of a concept is central to FCA. The way FCA looks at concepts is in line with the international standard ISO 704 that formulates the following definition: "A concept is considered to be a unit of thought constituted of two parts: its extent and its intent." *The extent* consists of all objects belonging to the formal concept, while *the intent* comprises all attributes shared by those objects. Let us illustrate the notion of formal concept of a formal context using the data in Table 1. For a set of objects $O \subseteq X$, the common features can be identified, written $O^{\uparrow}$, via:

$$A = O^{\uparrow} = \{y \in Y | \forall o \in O : (o, y) \in I\}$$

Take the attributes that describe paper 4 in Table 1, for example. By collecting all reports of this context that share these attributes, we get to a set $O \subseteq X$ consisting of papers 1 and 4. This set O of objects is closely connected to set A consisting of the attributes "browsing," "software," and "FCA".

$$O = A^{\downarrow} = \{x \in X | \forall a \in A : (x, a) \in I\}$$

That is, O is the set of all objects sharing all attributes of A, and A is the set of all attributes that are valid descriptions for all the objects contained in O. Each such pair (O, A) is called *a formal concept (or concept)* of the given context. The set $A = O^{\uparrow}$ is called the intent, while $O = A^{\downarrow}$ is called the extent of the formal concept (O, A).

There is a natural hierarchical ordering relation between the formal concepts of a given formal context that is called the subconcept/superconcept relation.

$$(O_1, A_1) \leq (O_2, A_2) \Leftrightarrow (O_1 \subseteq O_2 \Leftrightarrow A_2 \subseteq A_1)$$

A formal concept $C_1 = (O_1, A_1)$ is called a subconcept of a formal concept $C_2 = (O_2, A_2)$ (or equivalently, C_2 is called a superconcept of a formal concept C_1) if the extent of C_1 is a subset of the extent of C_2 (or equivalently, if the intent of C_1 is a superset of the intent of C_2). For example, the formal concept with intent "browsing," "software," "mining," and "FCA" is a subconcept of a formal concept with intent "browsing," "software," and "FCA." With reference to Table 1, the extent of the latter is composed of papers 1 and 4, while the extent of the former is composed of paper 1.

The set of all formal concepts of a formal context ordered by the subconcept/superconcept relation $\leq$ makes a complete lattice (called *the concept lattice* of the context), i.e. every subset of concepts has infimum (meet) and supremum (join) w.r.t. $\leq$. Concept lattices, like every ordered sets, can be visualized by line diagrams, where nodes stay for formal concepts and edges connect pairs of neighboring concept nodes, i.e. those that do not have any concept nodes between them. The line diagram in Figure 1, for example, is a compact representation of the concept lattice of the formal context abstracted from Table 1. The circles or nodes in this line diagram represent the formal concepts. The shaded boxes (upward) linked to a node represent the attributes used to name the formal concept. The non-shaded boxes (downward) linked to the node represent the objects used to name the formal concept. The information contained in the formal context of Table 1 can be distilled from the line diagram in Figure 1 by applying the following reading rule: An object "g" is described by an attribute "m" if and only if there is an ascending path from the node named by "g" to the node named by "m."
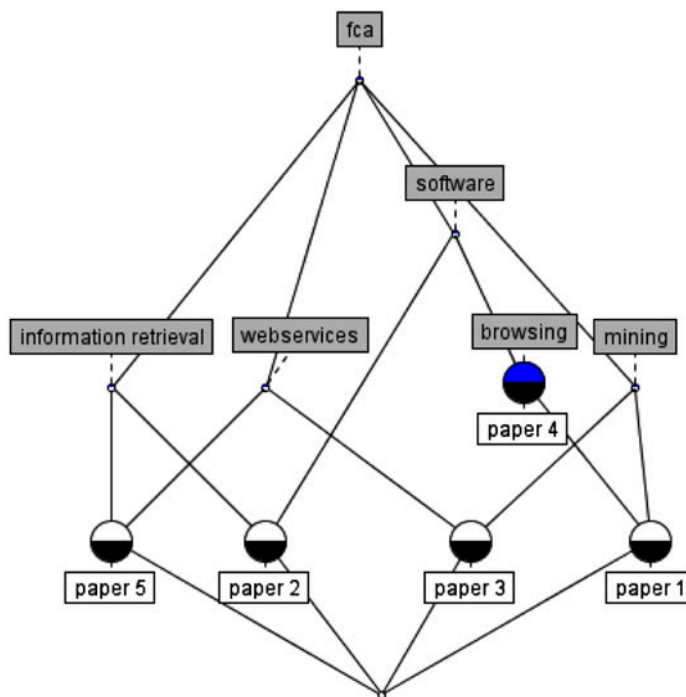
Figure 1. Line diagram corresponding to the context from Table 1.

For example, paper 1 is described by the attributes "browsing," "software," "mining," and "FCA."

Retrieving the extent of a formal concept from a line diagram such as the one in Figure 1 implies collecting all objects on all paths leading down from the corresponding node. To retrieve the intent of a formal concept, one traces all paths leading up from the corresponding node in order to collect all attributes. Note that a concept is a subconcept of all concepts that can be reached by travelling upward. The top and bottom concepts in the lattice are special. The top concept contains all objects in its extent. The bottom concept contains all attributes in its intent. This concept will inherit all attributes associated with these superconcepts.

2.2. *FCA software*

We developed a knowledge-browsing environment Concept Relation Discovery and Innovation Enabling Technology (CORDIET, Elzinga 2011; Poelmans et al. 2012b) to support our literature analysis. One of the central components of our text analysis environment is the thesaurus containing the collection of terms describing the different research topics. The initial thesaurus was constructed based on expert prior knowledge and was incrementally improved by analyzing the concept gaps and anomalies in the resulting lattices. The thesaurus is a layered structure containing multiple abstraction levels. The first and finest level of granularity contains the search terms of which most are grouped together based on their semantical meaning to form the term clusters at the second level of granularity.

The papers that were downloaded from the World Wide Web (WWW) were all formatted in pdf. These pdf files were converted to ordinary text, and the abstract, title, and keywords

were extracted. The open source tool Lucene was used to index the extracted parts of the papers using the thesaurus. The result was a cross table describing the relationships between the papers and the term clusters or research topics from the thesaurus. This cross table was used as a basis to generate the lattices.

3. Dataset

This Systematic Literature Review has been carried out by considering a total of 1072 papers related to FCA published between 2003 and 2011 in the literature and extracted from the most relevant scientific sources. The sources that were used in the search for primary studies contain the work published in those journals, conferences, and workshops that are of recognized quality within the research community. These sources are as follows:

- IEEE Computer Society.
- ACM Digital Library.
- Sciencedirect.
- Springerlink.
- EBSCOhost.
- Google Scholar.
- Conference repositories: ICFCA, ICCS and CLA conference.

Other important sources, such as DBLP or CiteSeer, were not explicitly included since they were indexed by some of the mentioned sources (e.g. Google Scholar). In the selected sources, we used various search strings including "FCA," "concept lattices," "Temporal Concept Analysis." To identify the major categories for the literature survey, we also took into account the number of citations of the FCA papers at CiteseerX.

4. Prominent application domains

The 1072 papers are grouped together according to a number of features within the scope of FCA research. We visualized the papers using concept lattices, which facilitate our exploration and analysis of the literature. The lattice in Figure 2 contains 7 categories under which the majority of the 1072 FCA papers can be categorized. Knowledge discovery is the most popular FCA research theme covering 23% of the papers and was analyzed in detail in the survey Poelmans et al. (2013a, 2013b). Recently, improving the scalability of FCA to larger and complex datasets emerged as a new research topic covering 9% of the 1072 FCA papers. In particular, we note that more than one-third of the papers dedicated to this topic work on issues in the Knowledge Discovery in Databases (KDD) domain. Scalability was also discussed in detail in Poelmans et al. (2013a). Another important research topic in the FCA community is information retrieval covering 13% of the papers. Thirty-si of the papers on information retrieval describe a combination with a KDD approach, and in 27, IR papers authors make use of ontologies. Fifteen papers on Information Retrieval deal with the retrieval of software structures such as software components. The FCA papers on information retrieval are discussed in detail by Poelmans et al. (2012). In 12% of the FCA papers, FCA is used in combination with ontologies or for ontology engineering. FCA research on ontology engineering was discussed in the survey (Poelmans et al. 2013b). Another important topic is using FCA in software engineering (13%), which was also discussed in Poelmans et al. (2013b). Finally,
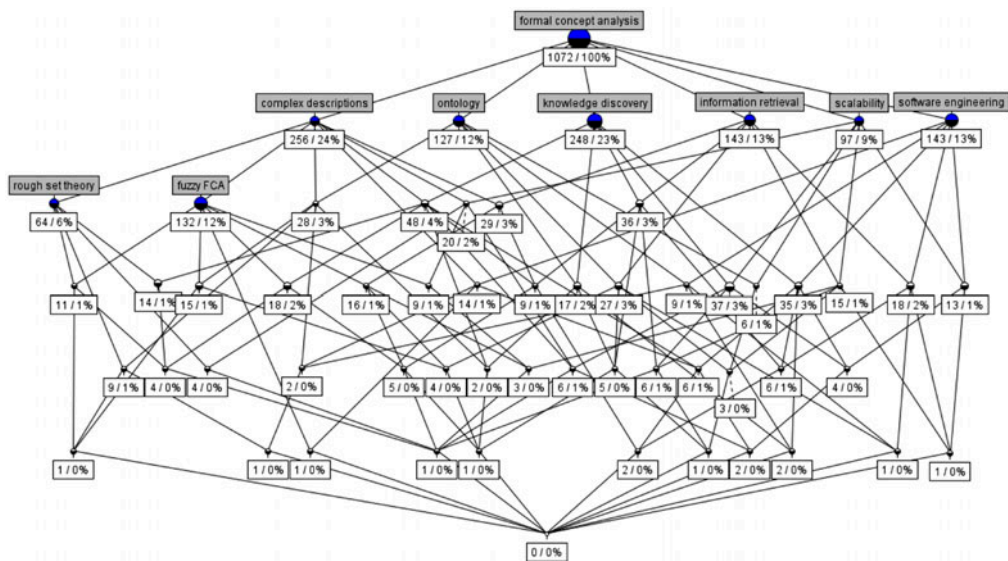
Figure 2. Lattice diagram containing 1072 papers on FCA with attributes describing major research subdomains.

in 24% of all papers, complex descriptions for FCA were investigated, see Poelmans et al. (2013a) for a detailed overview.

5. FCA with fuzzy attributes

In the basic setting of FCA, attributes are binary, i.e. table entries are 1 or 0 according to whether an object has the attribute or not. If the attributes under consideration are fuzzy (like "tall," "small"), each table entry contains a truth degree to which an attribute applies to an object. Fuzzy set theory was introduced by Zadeh (1965). Contrary to classical logic, fuzzy logic uses a scale L of truth degrees, and the most favorite choice is [0, 1] or some subchain of [0, 1]. This enables to consider intermediate truth degrees of propositions, e.g. "object x has attribute y" has a truth degree 0.8 indicating that the proposition is almost true. In addition to L, one has to pick an appropriate collection of logical connectives. The structure of truth degrees endowed by logical connectives is represented by a complete residuated lattice $\boldsymbol{L} = \langle L, \wedge, \vee, \otimes, \rightarrow, 0, 1 \rangle$ such that:

- $\langle L, \wedge, \vee, 0, 1 \rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively,
- $(L, \otimes, 1)$ is a commutative monoid, i.e. $\otimes$ is commutative, associative and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$,
- $\otimes$ and $\rightarrow$ satisfy the adjointness property $a \otimes b \leq c \Leftrightarrow a \leq b \rightarrow c$; for each a, b, $c \in L$,
- $\otimes$ and $\rightarrow$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication",
- By $\boldsymbol{L}^U$ (or L^U), we denote the collection of all fuzzy sets in a universe U, i.e. mappings A of U to L. For $A \in L^U$ and $a \in L$, a set ${}^aA = \{u \in U | A(u) \geq a\}$ is called an a-cut of A.

Fuzzy concept lattices were first introduced by Burusco and Fuentes-Gonzalez (1994), however, their theory had its limitations since they did not use residuated lattices and merely introduced some basic notions. Belohlavek and Pollandt further developed the basic notions of FCA with fuzzy attributes independently, so, there are two seminal works Belohlavek (1998) and Pollandt (1997) on this topic, e.g. in Belohlavek, Funiokova, and Vychodil (2005). The paper Belohlavek (1998) focuses on studying similarity in fuzzy concept lattices rather than introducing basic notions. In Belohlavek (2001b), the author studies fuzzy closure operators that are along with fuzzy Galois connections, the basic structures behind fuzzy concept lattices. These results have been later used many times in different papers on the topic. In Belohlavek (2001a) the author also shows a relationship between fuzzy Galois connections and ordinary Galois connections. He uses this result to show how a fuzzy concept lattice may be regarded as an ordinary concept lattice. Belohlavek and Vychodil (2005a) present a survey and comparison of the different approaches to fuzzy concept lattices which were elaborated till that time. Belohlavek and Vychodil (2006a) give an overview of recent developments concerning attribute implications in a fuzzy setting. The recent computational issues concerning algorithms for finding fuzzy concepts are discussed in the paper by Belohlavek et al. (2010).

The following definition of a formal fuzzy concept lattice was introduced in the study of Belohlavek (1999) and Belohlavek et al. (2005a). A fuzzy formal context (*L*-context) is a triple $\langle X, Y, I \rangle$, where X is the set of objects, Y is the set of attributes and $I : X \times Y \rightarrow L$ is a fuzzy relation (*L*-relation) between X and Y. A truth degree $I(x, y) \in L$ is assigned to each pair (x, y), Where $x \in X, y \in Y$ and L is the set of values of a complete residuated lattice $\boldsymbol{L}$. The element $I(x, y)$ is interpreted as the degree to which attribute y applies to object x. Fuzzy sets $A \in L^x$ and $B \in L^y$ are mapped to fuzzy sets $A^\uparrow \in L^y, B^\downarrow \in L^x$ according to (Belohlavek 1999).

$$A^\uparrow(y) = \underset{x \in X}{\Lambda}(A(x) \rightarrow I(x, y))$$

$$B^\downarrow(x) = \underset{y \in Y}{\Lambda}(B(y) \rightarrow I(x, y))$$

for $y \in Y$ and $x \in X$. A formal fuzzy concept $\langle A, B \rangle$ consists of a fuzzy set A of objects (extent of the concept) and a fuzzy set B of attributes (intent of the concept) such that $A^\uparrow = B$ and $B^\downarrow = A$. The set of all formal fuzzy concepts is $\mathfrak{B}(X, Y, I) = \{\langle A, B \rangle | \ A^\uparrow = B, \ B^\downarrow = A\}$. We also
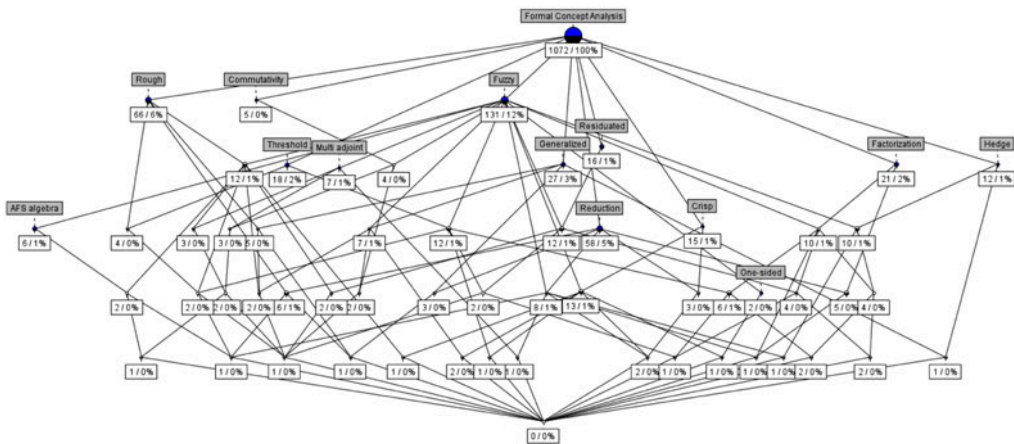

Figure 3. Lattice diagram describing the papers on FCA and fuzzy theory.

define ≤ which models the subconcept/superconcept hierarchy in $\mathfrak{B}(X, Y, I)$: $\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle \Leftrightarrow A_1 \subseteq A_2 \Leftrightarrow B_2 \subseteq B_1$ for $\langle A_1, B_1 \rangle$, $\langle A_2, B_2 \rangle \in \mathfrak{B}(X, Y, I)$. $\langle \mathfrak{B}(X, Y, I), \leq \rangle$, i.e. $\mathfrak{B}(X, Y, I)$ equipped with relation ≤ is a complete lattice w.r.t. ≤.

Figure 3 shows a lattice diagram with several prominent research topics within the FCA with fuzzy attributes research community. Section 5.1 discusses the different approaches to FCA with fuzzy attributes. Section 5.2 describes the papers on parameterized fuzzy concept lattice approaches. Section 5.3 discusses some of the variations on traditional FCA with fuzzy attributes theory. Section 5.4 covers fuzzy attribute logic, implications and inference. Section 5.5 discusses research on complementing FCA with fuzzy clustering. Section 5.6 covers the papers on matrix factorization, factor lattices and their implications for FCA with fuzzy attributes. The papers on using FCA with fuzzy attributes in KDD are covered in Section 5.7. The papers on using FCA with fuzzy attributes in IR are discussed in Section 5.8. The papers on using FCA with fuzzy attributes in ontology engineering are discussed in Section 5.9.

5.1. Alternative approaches to embed fuzzy logic into FCA

Several alternative approaches were proposed to the fuzzy concept lattice theory introduced in the previous section. In this section, we will survey the most important theories that were presented for combining FCA with fuzzy logic. Other formulations such as proto-fuzzy concepts, are discussed in the studies described by Kridlo and Krajci (2008), Pankratieva and Kuznetsov (2010, 2012).

5.1.1. Fuzzy concept lattice with non-commutative conjunction

Georgescu and Popescu (2002, 2003, 2004) defined the fuzzy concept lattice $(L, \vee, \wedge, \otimes, \rightarrow, \Rightarrow, 0, 1)$ where fuzzy logic with a noncommutative conjunction $\otimes$ is used instead of a commutative conjunction. They argue that omitting this commutativity requirement is necessary in situations where the order between terms of the conjunction counts, thus making the theory suitable for temporal data representation. In this situation the Galois connection will consist of two pairs of functions, namely $\uparrow, \Uparrow: L^X \rightarrow L^Y$ and $\downarrow, \Downarrow: L^Y \rightarrow L^X$ each function being in a symmetric situation to his pair. The authors also prove that any concept lattice for noncommutative fuzzy logic can be interpreted in their framework of generalized concept lattices with noncommutative conjunction.

5.1.2. One-sided fuzzy concept lattice

Yahia and Jaoua (2001) and Krajci (2003) independently developed the one-sided fuzzy concept lattice theory that has crisp sets as extents of concepts and fuzzy sets as intents of concepts. The authors introduce two mapping operators:

- f: $2^X \rightarrow L^Y$ for which $f(A)(y) = \Lambda_{x \in A} I(x, y)$, where $A \subseteq X$ is a set of objects and $f(A) \in L^Y$ is a fuzzy set of attributes,
- $h: L^Y \rightarrow 2^X$ for which $h(B) = \{x \in X \mid$ for each $y \in Y: B(y) \leq I(x, y)\}$, where $B \in L^Y$ is a fuzzy set of attributes and $h(B) \in X$ is a set of objects.

The authors proved that $\mathfrak{B}_{f,h}(X, Y, I) = \{\langle A, B \rangle \in 2^X \times L^Y \mid f(A) = B, h(B) = A\}$, together with partial order ≤ as defined in the introduction of Section 5 forms a complete lattice.

5.1.3. Crisply generated concepts

Another approach is to restrict the extracted fuzzy formal concepts to those that are crisply generated as described by Belohlavek, Sklenar, and Zacpal (2005). A formal fuzzy concept $\langle A, B \rangle \in \mathfrak{B}(X, Y, I)$ is crisply generated if there exists a crisp subset $B_c \subseteq Y$ such that $A = B_c^{\downarrow}$ (thus, $B = B_c^{\downarrow\uparrow}$). The lattice of crisply generated fuzzy concepts $B_c(X, Y, I) = \{ \langle A, B \rangle \in \mathfrak{B}(X, Y, I) |$ there exists $B_c \subseteq Y : A = B_c^{\downarrow} \}$ is isomorphic to the one-sided fuzzy concept lattice $\mathfrak{B}_{f,h}(X, Y, I)$ with fuzzy extents and crisp intents. Also for the corresponding concepts $\langle A, B \rangle \in \mathfrak{B}_{f,h}(X, Y, I)$ and $\langle C, D \rangle \in \mathfrak{B}_c(X, Y, I)$ there is an isomorphism such that $A = C$ and $B = D^{\downarrow\uparrow}$.

5.1.4. Generalized concept lattice

Krajci (2005a) defined generalized concept lattices, which use three sets of truth degrees, i.e. a set L_x for the objects, a set L_y for the attributes and L for the degrees to which objects have attributes. Generalized concept lattices were proposed to form a common platform for (Belohlavek 2004; Pollandt 1997), and the one-sided fuzzy concept lattices introduced by (Belohlavek 2005; Yahia 2001; Krajci 2003). Krajci (2005b) showed that generalized concept lattices embed some other approaches like the concept lattice with hedges (see Section 5.2). Medina, Ojeda-Aciego, and Ruiz-Calvino (2008) shows how the framework of generalized concept lattices can represent the concept lattice of Georgescu (see Section 5.1.1). It is isomorphic to a sublattice of the Cartesian product of two generalized concept lattices.

At the core, the two complete lattices are $\langle L_x, \leq \rangle$ and $\langle L_y, \leq \rangle$ and a poset $\langle L, \leq \rangle$. We use $\leq$ to refer to the partial order on L_x, L_y and L. The conjunction operator $\otimes: L_x \times L_y \to L$ is assumed to be increasing and left continuous in both arguments, i.e.:

- $a_1 \leq a_2 \Rightarrow a_1 \otimes b < a_2 \otimes b$,
- $b_1 \leq b_2 \Rightarrow a \otimes b_1 < a \otimes b_2$,
- if $a_j \otimes b \leq c$ for each $j \in J$ then $(\sup_{j \in J} a_j) \otimes b \leq c$,
- if $a \otimes b_j \leq c$ for each $j \in J$ then $a \otimes (\sup_{j \in J} b_j) \leq c$.

and this for each index set J and all $a, a_j \in L_x$, $b, b_j \in L_y$ and $c \in L$.

A fuzzy context is a tuple $\langle X, Y, I, \otimes \rangle$. Where X and Y are the object and attribute sets and $I \in L^{X \times Y}$ is an L-fuzzy relation. The arrow operators $\uparrow: L_X^X \to L_Y^Y$ and $\downarrow: L_Y^Y \to L_X^X$ are defined as:

- $A^{\uparrow}(y) = \sup\{ b \in L_Y | \forall x \in X: A(x) \otimes b \leq I(x, y) \}$,
- $B^{\downarrow}(x) = \sup\{ a \in L_X | \forall y \in Y: a \otimes B(y) \leq I(x, y) \}$.

where $A \in L_X^X$ and $B \in L_Y^Y$. The subset of $L_Y^Y \times L_X^X$ formed by the pairs $\langle A, B \rangle$ such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$ are the formal concepts. The set of all concepts $\mathfrak{B} = \{ \langle A, B \rangle \mid A^{\uparrow} = B$ and $B^{\downarrow} = A \}$ with partial order $\leq$ is a complete lattice, i.e. the generalized concept lattice for $\langle X, Y, I, \otimes \rangle$.

5.1.5. Multi-adjoint concept lattice

Multi-adjoint concept lattices were introduced by Medina, Ojeda-Aciego, and Ruiz-Calvino (2007), Medina, Ojeda-Aciego, and Ruiz-Calvino (2009). Adjoint triples, a.k.a. biresiduated structures (Medina 2004), are at the core of this approach. The multi-adjoint approach embeds

the fuzzy concept lattice approaches of Pollandt and Belohlavek, as well as the generalized concept lattice approach of Krajci (Medina, Ojeda-Aciego, and Ruiz-Calvino 2009).

We use ≤ to refer to the partial orders on P_1, P_2 and P_3. Let $\langle P_1, \leq \rangle$ and $\langle P_2, \leq \rangle$ and $\langle P_3, \leq \rangle$ be posets and $\& : P_1 \times P_2 \rightarrow P_3$, $\swarrow: P_3 \times P_2 \rightarrow P_1$, $\nwarrow: P_3 \times P_1 \rightarrow P_2$ be mappings, then $\langle \&, \swarrow, \nwarrow \rangle$ is an adjoint triple with respect to P_1, P_2 and P_3 if:

- $\&$ is order preserving in both arguments,
- $\swarrow$ and $\nwarrow$ are order preserving in the consequent and order reversing in the antecedent,
- $x \leq z \swarrow y \Leftrightarrow x \,\&\, y \leq z \Leftrightarrow y \leq z \nwarrow x$, where $x \in P_1$, $y \in P_2$ and $z \in P_3$ (adjoint property).

We use ≤ to refer to the partial orders on L_x, L_y and L. A multi-adjoint frame $\mathcal{L}$ consists of several adjoint triples:

$$\langle L_x, L_y, L, \leq, \leq, \leq, \&_1, \swarrow^1, \nwarrow_1, \ldots, \&_n, \swarrow^n, \nwarrow_n \rangle,$$

where $\langle L_x, \leq \rangle$ and $\langle L_y, \leq \rangle$ are complete lattices and $\langle L, \leq \rangle$ is a poset and $\langle \&_i, \swarrow^i, \nwarrow_i \rangle$ is an adjoint triple with respect to L_x, L_y, L for all $i = 1, \ldots, n$. Given a multi-adjoint frame, a context is a tuple $\langle X, Y, I, \sigma \rangle$ such that X (objects) and Y (attributes) are nonempty sets, I is a fuzzy L-relation $I : X \times Y \rightarrow L$ and $\sigma : Y \rightarrow \{1, \ldots, n\}$ is a mapping which associates any element in Y with a particular adjoint triple in the frame.

The arrow operators $\uparrow_\sigma: L_Y^Y \rightarrow L_X^X$ and $\downarrow_\sigma: L_X^X \rightarrow L_Y^Y$ are further defined as:

- $A^{\uparrow\sigma}(x) = \inf\{I(x, y) \swarrow^{\sigma(y)} A(y) \mid y \in Y\}$.
- $B^{\downarrow\sigma}(y) = \inf\{I(x, y) \nwarrow_{\sigma(x)} B(x) \mid x \in X\}$.

They generate a Galois connection and generalize the arrow operators in the L-fuzzy and generalized concept lattice approaches.

A concept is a pair $\langle A, B \rangle$ such that $A \in L_Y^Y$ and $B \in L_X^X$ and $A^{\uparrow\sigma} = B$ and $B^{\downarrow\sigma} = A$. The multi-adjoint concept lattice is the set $\mathfrak{B} = \{\langle A, B \rangle \mid A \in L_Y^Y, B \in L_X^X \text{ and } A^{\uparrow\sigma} = B, B^{\downarrow\sigma} = A\}$ in which the ordering is defined by $\langle a_1, b_1 \rangle \leq \langle a_2, b_2 \rangle \Leftrightarrow a_1 \leq a_2 \, (b_2 \leq b_1)$.

5.2. Parameterized approaches for fuzzy concept lattices

An issue that may arise while using fuzzy concept lattices is the potentially large number of concepts extracted from the data. Several parameterized approaches were introduced where the parameters control the number of extracted formal concepts.

5.2.1. Concept lattices with thresholds

Thresholds are parameters that can be used to control the number of extracted concepts. Starting from the definitions in the introduction of Section 5; by imposing threshold δ, which is an arbitrary truth degree of L, we can obtain a set $\delta_{A\uparrow} = \{y \mid A^\uparrow(y) \geq \delta\}$ containing only those attributes which belong to $A^\uparrow$ in a degree greater than or equal to δ. For $\delta = 1$, we obtain the approaches of Belohlavek, Sklenar, and Zacpal (2005), Belohlavek and Vychodil (2005b), Yahia and Jaoua (2001) and Krajci (2003). Elloumi et al. (2004) extended this approach to arbitrary δ, however, their extent- and intent-forming operators did not form a Galois connection. This was resolved in (Zhang et al. 2007), where the authors proposed new operators based on the idea of a threshold for general δ. Belohlavek (2007) noted that these

threshold-based operators are reducible to classical concept-forming operators. We will now describe the threshold approach of Zhang et al. (2007) building further on the definitions in the introduction of Section 5. For $A \in L^X$, $C \in 2^X$, $B \in L^Y$, $D \in 2^Y$, the authors define three additional pairs of operators $\langle *, * \rangle$, $\langle \Box, \Box \rangle$ and $\langle \triangle, \triangle \rangle$:

- $C^* = \{y \in Y \mid \inf_{x \in X} (C(x) \to I(x, y) \geq \delta\} \in 2^Y$
  $D^* = \{x \in X \mid \inf_{y \in Y} (D(y) \to I(x, y) \geq \delta\} \in 2^X$
- $C^{\Box}(y) = (\delta \to \inf_{x \in C} I(x, y)) \in L^Y$
  $B^{\Box} = \{x \in X \mid \inf_{y \in Y} (B(y) \to I(x, y) \geq \delta\} \in 2^X$
- $A^{\triangle} = \{y \in Y \mid \inf_{x \in X} (A(x) \to I(x, y) \geq \delta\} \in 2^Y$
  $D^{\triangle}(x) = (\delta \to \inf_{y \in D} I(x, y)) \in L^Y$

for each $x \in X$, $y \in Y$.

The authors prove that these operators form a Galois connection resulting in the following concept lattices:

- $\mathfrak{B}(X_*, Y_*, I) = \{\langle A, B \rangle \in 2^X \times 2^Y \mid A^* = B, B^* = A\}$, which equals the ordinary concept lattice $\mathfrak{B}(X, Y, {}^{\delta}I)$ where ${}^{\delta}I = \{\langle x, y \rangle \in X \times Y \mid I(x, y) \geq \delta\}$ (Belohlavek 2007).
- $\mathfrak{B}(X_{\triangle}, Y_{\triangle}, I) = \{\langle A, B \rangle \in L^X \times 2^Y \mid A^{\triangle} = B, B^{\triangle} = A\}$, which is a one-sided fuzzy concept lattice with fuzzy extents and crisp intents induced by $(X, Y, \delta \to I)$.
- $\mathfrak{B}(X_{\Box}, Y_{\Box}, I) = \{\langle A, B \rangle \in 2^X \times L^Y \mid A^{\Box} = B, B^{\Box} = A\}$, which is a one-sided fuzzy concept lattice with crisp extents and fuzzy intents.

Belohlavek, Outrata, and Vychodil (2006) show that the approach via thresholds can be seen as a particular case of the approach via hedges. For data with fuzzy attributes, the fuzzy concept lattice obtained with the operators in the study described by Zhang et al. (2007) is isomorphic to a fuzzy concept lattice with hedges induced from data containing shifts of the given fuzzy attributes (see also Section 5.3).

5.2.2. Concept lattices with hedges

Linguistic hedges are expressions such as "very," "more or less," and "extremely" (Zadeh 1972) and can be used in the context of FCA with fuzzy attributes to control the number of extracted concepts. These concept lattices with hedges were first introduced by Belohlavek and Vychodil (2005a, 2007) and later on generalized in Belohlavek and Vychodil (2012). Belohlavek and Vychodil (2012) show that the fuzzy concept lattices in Burusco and Fuentes-Gonzales (1994), Pollandt (1997), Belohlavek (2002b), Ben Yahia and Jaoua (2001) and Krajci (2003) are special cases of the concept lattices with hedges. A truth-stressing hedge * (further on called hedge) on L can be seen as a truth function of a unary logical connective such as "very" (Hájek 2001). It is a mapping on L satisfying: $1* = 1$, $a* \leq a$, $(a \to b)* \leq a* \to b*$ and $a** = a*$. Starting from a context $\langle X, Y, I \rangle$, suppose that for each object $x \in X$ a hedge *x on L is given and for each attribute $y \in Y$, a hedge *y on L is given. For fuzzy sets $A \in L^X$ and $B \in L^Y$, consider fuzzy sets $A^{\uparrow} \in L^Y$ and $B^{\downarrow} \in L^X$ defined as:

- $A^{\uparrow}(y) = \inf_{x \in X}(A(x)^{*x} \to I(x, y))$
- $B^{\downarrow}(x) = \inf_{y \in y}(B(y)^{*x} \to I(x, y))$

The set $\mathfrak{B}(X^{*X}, Y^{*Y}, I) = \{\langle A, B \rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\}$ is a fuzzy concept lattice with hedges, where *X is the collection of all *x's and *Y is the collection of all *y's. For $\langle A_1, B_1 \rangle$,

$\langle A_2, B_2 \rangle \in \mathfrak{B}(X^{*X}, Y^{*Y}, I)$ we define $\langle A_1, B_1 \rangle \leq \langle A_2, B_2 \rangle \Leftrightarrow A_1 \subseteq A_2$ $(B_2 \subseteq B_1)$. $\mathfrak{B}(X^{*X}, Y^{*Y}, I)$ in combination with $\leq$ was proven to be a complete lattice (Belohlavek and Vychodil 2012).

Krajci (2005b) showed that in case there is one hedge *1 for X and one hedge *2 for Y and *x = *1 for each $x \in X$ and *y = *2 for each $y \in Y$, (i.e. the originally introduced concept lattice with hedges, by Belohlavek and Vychodil (2005a, 2007) that each such concept lattice with hedges is isomorphic to some generalized concept lattice.

5.3. *Variations to traditional FCA with fuzzy attributes*

Belohlavek et al. (2007c) investigate the problem of approximating possibly infinite sets of solutions by finite sets of solutions. These infinite sets of solutions typically appear in constraint-based problems such as "find all collections in a given finite universe satisfying constraint C". In fuzzy setting, when collections are conceived as fuzzy sets, the set of all such collections may be infinite and computationally intractable when one uses the unit interval $[0, 1]$ as the scale of membership degrees. The authors propose to use a finite subset K of $[0, 1]$ which approximates $[0, 1]$ to a satisfactory degree and illustrate the idea on FCA. Since the extent to which "object o has property a" may be sometimes hard to assess precisely. Djouadi et al. (2009, 2010) use a subinterval from the scale L, rather than a precise value. Such formal contexts lead to interval-valued formal concepts. The authors provide a minimal set of requirements for interval-valued implications in order to fulfill the fuzzy closure properties of the resulting Galois connection. Secondly, a new approach based on a generalization of Gödel's implication is proposed for building the complete lattice of all interval-valued formal concepts. Zerarga and Djouadi (2012) investigate the possibilities of their approach for information retrieval applications.

Conceptual scaling (Ganter and Wille 1999) is a process of transformation of data tables with general attributes, e.g. nominal, ordinal, etc. to data tables with yes/no attributes. In this way, data tables with general attributes can be analyzed by means of FCA. Belohlavek et al. (2007b) present a new method of scaling, namely scaling of general attributes to fuzzy attributes. After such a scaling, the data can be analyzed by means of FCA with fuzzy attributes. Compared with ordinary scaling, the author argues that this scaling is less sensitive to how a user defines a scale that eliminates the arbitrariness of the user's definition of a scale. Belohlavek et al. (2008c) present necessary and sufficient conditions on input data for the output concept lattice to form a tree after one removes its last element, in particular, for input data with fuzzy attributes.

5.4. *Fuzzy attribute logic, implications and inference*

Belohlavek et al. (2006c) present fuzzy attribute logic (FAL) for reasoning about formulas describing particular attribute dependencies. The formulas are of the form $A \rightarrow B$ where A and B are collections of attributes. There formulas can be interpreted in two ways. First, in database tables with entries containing degrees to which objects have attributes. Second, in database tables where each domain is equipped with a similarity relation assigning a degree of similarity to any pair of domain elements. If the scale contains only two degrees, 0 and 1, two well-known calculi become particular cases of FAL With the first interpretation, FAL coincides with attribute logic used in FCA, with the second interpretation, the logic coincides with Armstrong system for reasoning about functional dependencies. Belohlavek et al. (2006d) focus on similarity related to attribute implications, i.e. rules $A \rightarrow B$ describing dependencies "each object which has all attributes from A has

also all attributes from *B*." The authors present several formulas for the estimation of similarity of outputs in terms of similarity of inputs. Fan et al. (2006) propose a new form of fuzzy concept lattice and two coherent fuzzy inference methods for this and three existing kinds of fuzzy concept lattices. The lower approximate fuzzy inference and the upper approximate fuzzy inference based on the fuzzy concept lattice are proposed, and the authors claim that the combined use of the two methods makes the fuzzy inference more precise.

5.5. *Combining FCA with fuzzy clustering*

Fuzzy clustering allows objects of a dataset to belong to several clusters simultaneously, with different degrees of membership. Sassi et al. (2007) use FCA for cluster quality evaluation, i.e. interpreting and distinguishing overlapping clusters and evaluating the quality of clusters. Nobuhara et al. (2006) present a hierarchical representation method for image/video databases based on FCA and fuzzy clustering. Fuzzy clustering is first performed to a vast amount of objects and the number of objects is reduced into suitable numbers for visualization. The method was confirmed to be helpful to do video clipping and grasp the whole structure of an image database. Kumar et al. (2009) propose a method based on Fuzzy K-means clustering for reducing the size of concept lattices. They demonstrate the usefulness of their method for information retrieval and information visualization.

5.6. *Factorization by similarity in FCA with fuzzy attributes*

Belohlavek et al. (2007a) describe how to obtain the factor lattice from a large fuzzy concept lattice by means of a similarity relation. The factor lattice contains fewer clusters than the original concept lattice but at the same time represents a reasonable approximation of the original lattice. The user can specify a similarity threshold and a smaller threshold leads to smaller factor lattices, i.e. more comprehensible but less accurate approximations of the original concept lattice. In Belohlavek et al. (2008b), this research is extended to fuzzy concept lattices with hedges. Shifts of fuzzy attributes play an important role in the efficient computation of a factorization by similarity of a fuzzy concept lattice (Belohlavek 2000; Belohlavek et al. 2004a).

5.7. *FCA with fuzzy attributes in Knowledge Discovery and DM*

KDD and DM is an interdisciplinary research area focusing upon methodologies for extracting useful knowledge from data. In the past, the focus was on developing fully automated tools and techniques that extract new knowledge from data. Unfortunately, these techniques allowed almost no interaction between the human actor and the tool and failed at incorporating valuable expert knowledge into the discovery process (Keim 2002), which is needed to go beyond uncovering the fool's gold. These techniques assume a clear definition of the concepts available in the underlying data that are often not the case. Visual data exploration (Eidenberger 2004) and visual analytics (Thomas and Cook 2005) are especially useful when little is known about the data and exploration goals are vague. Since the user is directly involved in the exploration process, shifting and adjusting the exploration goals is immediately done if necessary.

In Conceptual Knowledge Processing (CKP), the focus lies on developing methods for processing information and knowledge, which stimulate conscious reflection, discursive

argumentation and human communication (Wille 2006). The word "conceptual" underlines the constitutive role of the thinking, arguing and communicating human being and the term "processing" refers to the process in which something is gained which may be knowledge. An important subfield of CKP is Conceptual Knowledge Discovery (Stumme 2003). FCA is particularly suited for exploratory data analysis because of its human-centeredness (Hereth et al. 2003). The generation of knowledge is promoted by the FCA representation that makes the inherent logical structure of the information transparent. The philosophical and mathematical origins of using FCA for knowledge discovery have been briefly summarized in the study described by Priss (2006). The systems TOSCANA (Vogt et al. 1994), Concept Explorer (Yevtushenko 2000), FcaStone (Priss 2008), InClose (Andrews 2009), ToscanaJ (Becker et al. 2002), Galicia (Valtchev et al. 2008), CORDIET (Poelmans et al. 2012) have been used as knowledge discovery tools in various research and commercial projects.

Sixteen percent of the papers use FCA with fuzzy attributes for knowledge discovery. Sklenar et al. (2005) used FCA to evaluate epidemiological questionnaire physical activity data to find dependencies between demographic data and degree of physical activity. Belohlavek et al. (2007, 2011) build further on the work of Sklenar et al. (2005) and Sigmund et al. (2005) by aggregating respondents and using fuzzy values to indicate the relative frequency of the attributes in the aggregated objects. Bertaux et al. (2009) describe a method to identify ecological traits of species based on the analysis of their biological characteristics. The complex structure of the dataset is formalized as a fuzzy many-valued context and transformed into a binary context through histogram scaling. The core of the method relied on the construction and interpretation of formal concepts. The concepts were interpreted by a hydrobiologist, leading to a set of ecological traits that were inserted in the original context.

Efficient service management is an important task for service execution and service composition. Classification and semantic annotation of services are important challenges in service-centric software engineering. It is challenging to effectively manage and search the web services satisfying the requestor's requirements. Fenza et al. (2008, 2009) present a system which uses FCA with fuzzy attributes for supporting the user in the discovery of semantic web services. This system is divided into lower and upper layers. In the lower layer, the semantic descriptions of the web services are transformed into fuzzy multisets. This description is an OWL-S document that sketches the capabilities of the service. Using Fuzzy C-Means clustering, the web services are grouped in fuzzy clusters. Fuzzy matchmaking is performed to retrieve those services that are approximate replies to the input request. In the upper layer, a fuzzy formal context is used to represent the prototypes (i.e. representative semantic web services for the clusters retrieved in the lower layer) and the given ontological concepts that are present or not. Navigating in the lattice, the user can then discover terminology associated with the web resources and may use it to generate an appropriate service request. Geng et al. (2008) took emails as objects, keywords as attributes, and built a lattice to identify email topics. A fuzzy membership value was used during postprocessing to extract the concepts that best represent a single topic. Chou et al. (2008) used web APIs as objects and tags describing these APIs as attributes. They use FCA with fuzzy attributes to identify the best candidates for fulfilling a service request of a user. Zhang et al. (2007b) discuss the extraction of fuzzy linguistic summaries from a continuous information system. They use FCA in combination with degree theory to obtain these fuzzy linguistic summaries (Zhang and Liu 1996). Feng et al. (2008) perform schema matching by mapping elements of a schema S onto elements of a schema T by using a fuzzy concept lattice, Naïve Bayes and a structural similarity measure.

5.8. *FCA with fuzzy attributes in Information retrieval*

According to Manning et al. (2008), information retrieval (IR) is finding material (usually documents) of an unstructured nature (usually text) that satisfies an information need from within large collections (usually stored on computers). The field of information retrieval also covers supporting users in browsing or filtering document collections or further processing a set of retrieved documents. Given a set of documents, clustering is the task of coming up with a good grouping of the documents based on their contents. Given a set of topics, standing information needs, or other categories, classification is the task of deciding which classes, each of a set of documents belongs to.

The first attempts to use lattices for information retrieval are summarized in the study of Priss (2000), but none of them resulted in practical implementations. Godin et al. (1989) developed a textual information retrieval system based on document-term lattices but without graphical representations of the lattices. The authors also compared the system's performance to that of Boolean queries and found that it was similar to and even better than hierarchical classification (Godin et al. 1993). They also worked on software component retrieval (Mili et al. 1997). In Carpineto et al. (2004a), their work on information retrieval was summarized. They first argue that lattices can be used to make suggestions for query enlargement in cases where too few documents are retrieved and for query refinement in cases where too many documents are retrieved. Second, lattices can support an integration of querying and navigation (or browsing). An initial query identifies a start node in a document-term lattice. Users can then navigate to related nodes. Further, queries are then used to "prune" a document-term lattice to help users focus their search (Carpineto et al. 1996b). For many purposes, some extra facilities are needed: processing large document collections quickly, allow more flexible matching operations and allow ranked retrieval.

In traditional information retrieval, queries have not taken into account imprecision and retrieve only elements that precisely match to the given Boolean query. That is, an element belongs to the result if the query is true for this element; otherwise, no answers are returned to the user. In 9% of the IR papers, authors make use of FCA with fuzzy attributes. In Quan, Hui, and Cao (2004), FCA with fuzzy attributes is used for conceptual clustering in a citation-database document retrieval system. Using fuzzy logic in combination with FCA, a fuzzy concept lattice is constructed on which a fuzzy conceptual clustering technique is performed. Fuzzy queries can then be performed for document retrieval. Butka et al. (2008) describe an approach using FCA with fuzzy attributes for the creation of an ontology. The starting set of documents (or objects) is decomposed into smaller sets of similar documents with the use of the Growing Hierarchical Self Organizing Map clustering algorithm. Then one concept lattice is built upon every cluster using FCA with fuzzy attributes. The resulting models are combined to a hierarchy of concept lattices using agglomerative clustering. Hassine et al. (2008) also propose a method based on FCA and fuzzy logic that permits the flexible modeling and querying of a database. Chettaoui et al. (2008) use FCA with fuzzy attributes to deal with empty answers for fuzzy queries. Fuzzy querying processing based on Galois lattices allows detecting the reasons of empty answers by providing the subqueries that are responsible for the failure. Hachani et al. (2009) build further on this work and make use of FCA and fuzzy logic to interactively with the user explain the reasons of the failure of the query (i.e. no answer is returned to the user) and to propose the nearest answers. These answers that are in the neighborhood of the user's original query could be used to serve the user's need. Zhang et al. (2008) present a method based on FCA with fuzzy attributes for query enlargement.

5.9. *FCA with fuzzy attributes in Ontology Engineering*

Ontologies were introduced as a means of formally representing knowledge. Their purpose is to model a shared understanding of the reality as perceived by some individuals in order to support knowledge intensive applications (Gruber 2009). Ontology typically consists of individuals or objects, classes, attributes, relations between individuals and classes or other individuals, function terms, rules, axioms, restrictions and events. The set of objects that can be represented is called the universe of discourse. The axioms are assertions in a logical form that together comprise the overall theory that the ontology describes in its domain of application. Ontologies are typically encoded using ontology languages, such as the Ontology Web Language (OWL). Whereas ontologies often use tree-like representations for modeling the world, FCA has the benefit of partial order lattice representation that has a larger expressive power (Christopher 1965). A key objective of the semantic web is to provide machine interpretable descriptions of web services so that other software agents can use them without having any prior "built-in" knowledge about how to invoke them. Ontologies play a prominent role in the semantic web where they provide semantic information for assisting communication among heterogeneous information repositories.

Quan et al.(2004) use FCA with fuzzy attributes for the automatic generation of ontologies. These ontologies are used to support the Scholarly Semantic Web, which is used for sharing, reuse and management of scholarly information. Quan et al. (2006) propose a method based on FCA with fuzzy attributes, which they call Fuzzy Ontology Generation Framework, to automatically generate an ontology. In Quan et al. (2006), the authors apply this method to build an ontology that can be used in a web-based help-desk application. In a case study, they analyzed 9000 records stored in the customer service support database of a machine manufacturing company. Each of these records contains the description of a machine failure reported by a customer and proposed remedies to resolve the problem. Keywords, such as certain machine parts, can be extracted from this piece of text. A fuzzy formal context will then relate these machine failure cases (objects) with keywords explaining the nature of the failure (attributes) through a membership value indicating for example the possibility of a machine part being involved in a failure case. From this context, a fuzzy fault concept lattice is created and a fault concept hierarchy is built from it by applying fuzzy clustering of the concepts. This hierarchy is then translated into an ontology that can be used to automatically suggest actions which can be undertaken to resolve a failure. Zhou, Liu, and Zhao (2007) present an approach that is similar to this work. The authors derive a concept hierarchy from a fuzzy concept lattice which they applied as classification instrument on 13 datasets of UCI Machine Learning Repository. Maio et al. (2009, 2012) use FCA with fuzzy attributes to build an ontology for automatically classifying RSS feeds. From these feeds, relevant keywords are extracted and a fuzzy context is built. The concept lattice constructed from this context is then automatically translated to an OWL ontology. A validation experiment was performed with 443 feeds from the OpenLearn project, containing the Open University's course materials, which were manually arranged in categories based on educational subjects. The authors achieved good performance with their method, 87% of feeds were categorized coherently with OpenLearn's manual classification.

6. Rough set theory

Rough Set Theory (RST) was introduced by Pawlak (1982, 1985, 1991) and is a mathematical technique to deal with uncertainty and imperfect knowledge. In rough set theory, the data for analysis consists of universe *U*. Objects characterized by the same properties are indiscernible

(similar) in view of the available information about them. By modeling indiscernibility as an equivalence relation, $E \subseteq U \times U$ one can partition a finite universe of objects into pair wise disjoint subsets denoted by U/E. The partition provides a granulated view of the universe. An equivalence class is considered as a whole, instead of many individuals. For an object $x \in U$, the equivalence class containing x is given by $[x]_E = \{y \in U| \, xEy\}$. Objects in $[x]_E$ are indistinguishable from x. The empty set, equivalence classes and unions of equivalence classes form a system of definable subsets under discernibility. It is a σ-algebra $\sigma(U/E) \subseteq 2^U$ with basis U/E where 2^U is the power set of U. All subsets not in the system are consequently approximated through definable sets. Various definitions of rough set approximations have been proposed including the subsystem-based, granule-based and element-based formulation (Yao and Chen 2005). In this section we introduce the subsystem-based formulation. In an approximation space $apr = (U, E)$, a pair of approximation operators $\overline{(\cdot)} : 2^U \to 2^U$ and \underline $(\cdot) : 2^U \to 2^U$, is defined by

$$\underline{A} = \cup \{X | X \in \sigma(U/E), X \subseteq A\}$$

$$\overline{A} = \cap \{X | X \in \sigma(U/E), A \subseteq X\}$$

The lower approximation $\underline{A} \in \sigma(U/E)$ is the greatest definable set contained in A, and the upper approximation $\overline{A} \in \sigma(U/E)$ is the largest definable set containing A. Any set of all indiscernible (similar) objects is called an elementary set (neighborhood) and forms a basic granule (atom) of knowledge about the universe and $apr = (U, E)$ is called an approximation space. Any union of elementary sets is a crisp (precise) set; otherwise, the set is rough (imprecise, vague). Each rough set has boundary-line cases, i.e. objects that cannot be classified with certainty as either members of the set or its complement. Crisp sets have no boundary-line elements at all. Boundary-line cases cannot be properly classified by employing the available knowledge.
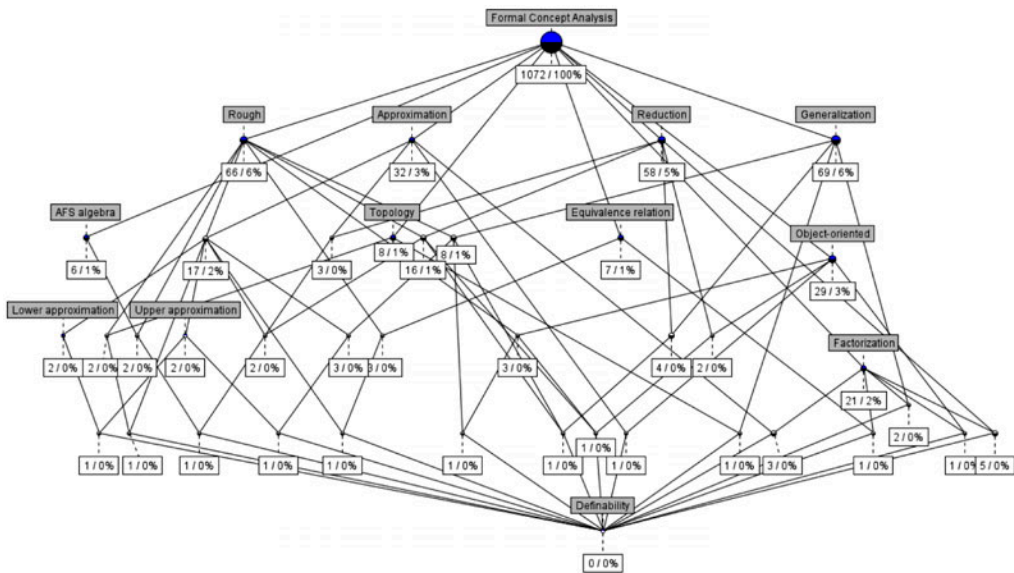

Figure 4. Lattice diagram describing the papers on FCA and rough set theory.

Vague concepts (in contrast to precise concepts) cannot be characterized in terms of information about their elements. In other words, given an arbitrary subset $A \subseteq U$ of the universe of objects, it may not be the extent of a formal concept. This subset can be seen as an undefinable set of objects and can be approximated by definable sets of objects, namely extents of formal concepts. Any vague concept is replaced by a pair of precise concepts, called the lower and upper approximation of the vague concept. The lower approximation consists of all objects that surely belong to the concept and the upper approximation contains all objects which possibly belong to the concept. The difference between the lower and upper approximation constitutes the boundary region of the vague concept.

Figure 4 gives an overview of several prominent research topics in the FCA and rough set theory community in the form of a lattice diagram. In Section 6.1, we discuss the papers on combining the theory of FCA with rough set theory. Section 6.2 covers the different types of generalizations of traditional RST that were introduced over the years and the link with FCA. Section 6.3 describes the papers on combining FCA with fuzzy attributes with RST. Section 6.4 covers the papers on attribute reduction in concept lattices using RST.

6.1. Combining FCA with rough set theory

Many efforts have been made to combine FCA and rough set theory (Yao 2004a). This combination is typically referred to as Rough Formal Concept Analysis (RFCA). In RFCA $\sigma(U/E)$ is replaced by lattice $\boldsymbol{L}$ and definable sets of objects by extents of formal concepts. The extents of the resulting two concepts are the approximations of A. For a set of objects $A \subseteq U$, its lower and upper approximations are defined by

$$\underline{l}(A) = \tau(\sigma(\cup\{X|(X,Y) \in L, X \subseteq A\})))$$

$$\overline{l}(A) = \cap\{X|(X,Y) \in L, A \subseteq X\}$$

The lower approximation of a set of objects A is the extent of $(\underline{l}(A), \sigma(\underline{l}(A)))$ and the upper approximation is the extent of the formal concept $(\overline{l}(A), \sigma(\overline{l}(A)))$. The concept $(\overline{l}(A), \sigma(\underline{l}(A)))$, is the supremum of concepts where extents are subsets of A and $(\overline{l}(A), \sigma(\underline{l}(A)))$ is the infimum of those concepts where extents are superset of A. For the other RST formulations a similar description can be given for the combination of FCA and RST.

Gediga and Duntsch (2002) introduced a lattice construction algorithm, which makes use of approximation operators. The extent and intent of a formal concept can be viewed as two systems of definable sets of objects and attributes respectively (Yao 2004a, 2004b). The approximation operators can then be formulated with respect to these systems and introduced into FCA. Deogun and Jiang (2005) evaluate three approaches for concept approximation. Concept approximation is to find the best or closest concepts to approximate a pair of objects and features. Under the circumstances, a set of objects and a set of attributes do not form a concept, concept approximation will give the best solution. Kent (1996) uses an equivalence relation on the set of objects. With respect to the formal context, a pair of upper and lower context approximations is defined. The two context approximations are then used to define a pair of lower and upper approximations of concepts. The other approach is based on the system of definable concepts in the concept lattice (Hu et al. 2001; Saquer et al. 1999a, 1999b). Saquer et al. (1999a, 1999b) studied approximations of a set of objects, a set of properties, and a pair of a set of objects and a set of properties, based on the formal concepts of a concept lattice. For example, given a set of objects, the authors tried to approximate the set by formal concepts whose extents approximate the set. An equivalence relation is introduced on

the set of objects from a formal context, which leads to rough set approximations. Their formulation is flawed since an equivalence class is not necessarily the extent of a formal concept. The union of extents of a family of formal concepts may not be the extent of a formal concept. Hu et al. (2001) suggested an alternative formulation to ensure that approximations are indeed formal concepts. Instead of an equivalence relation, they defined a partial order on the set of objects. Unfortunately, their definition of lower approximation had the same shortcoming as Saquer's.

The notion of approximation operators can be defined based on two universes linked by a binary relation (Yao 2004a). Based on the common notion of definability, Yao and Chen (2005) propose a framework for using rough set approximations in FCA. Comparative examination of RST and FCA shows that each of them deals with a particular type of definability. While FCA focuses on sets of objects that can be defined by conjunction of properties, RST focuses on disjunction of properties. An arbitrary concept is approximated from below and above by two definable concepts. The author shows that the problem with existing studies can be solved by a clear separation of two systems, the concept lattice and the system of extents of formal concepts. The two systems give rise to two different types of approximation. Yao and Chen (2006) further compare and combine both theories based on definability. There is a close connection between definability and approximation. A definable set of the universe of objects is definable if and only if its lower approximation is equal to its upper approximation.

Grabowski and Schwarzweller (2004) investigate the facilities of the Mizar system concerning extending and combining theories based on structure and attribute definitions and as an example, the authors consider the formation of rough concept analysis out of FCA and RST. Pagliani (2006) introduced a framework for comparing and combining FCA and rough set systems. Jiang et al. (2006) introduced rough-valued contexts into FCA and defined how to obtain formal concepts from these extended contexts. Lai and Zhang (2009) show that the expressive power of concept lattices based on rough set theory is weaker than that of concept lattices based on FCA. Xu, Xu, and Zhang (2007) present the notion of information concept lattice for which some properties are given. They also present a reduction theory for information concept lattices and compare the information concept lattice with rough set theory and concept lattices. Shao and Zhang (2007) combine RST and FCA and use this combination to define information granules in information systems. The authors show all the information granules form a complete lattice and present approaches for attribute reduction and rule acquisition for information granularity lattices.

6.2. Generalizations of rough set theory

Classical RST is developed based on an equivalence (indiscernibility) relation on a universe of objects. The equivalence relation is a stringent condition that limits the application domain of the rough set model. Generalized formulation has been proposed by using a binary relation on two universes, one is the set of objects and the other the set of properties (Yao and Lin 1996; Gediga 2002). A binary relation on two universes is known as a formal context in FCA and serves as a common basis for RST and FCA. Many authors have generalized the notion of approximation operators. The notions of formal concept and lattice can be introduced in RST by constructing different types of formal concepts. Duntsch et al. (2003), following the study of modal logics, defined modal-style operators based on a binary relation and introduced the property-oriented concept lattice. The derivation operator of FCA is a polarity or sufficiency operator used in modal logics, and the rough set approximation operators are the necessity and possibility operators used in modal logics. Yao (2004b) uses the

formulation in terms of modal-style operators to gain more insights into RST and FCA and defined the object-oriented formal concept lattice. Chen et al. (2008) defines two *L*-rough approximation operators by an arbitrary *L*-relation, some of their properties and relation to Galois connection in FCA are investigated. The generalizations of the property-oriented concept lattice and the object-oriented concept lattice are obtained in *L*-rough sets.

One way to define approximation operators is called the subsystem-based formulation (Yao 2005). The elements of the subsystem are understood as definable or observable sets. Every subset of the universe is approximated from below and above by two sets in the subsystem. There are two basic restrictions of the standard Pawlak model. First, a single subsystem of the power set is used. Second, the subsystem is closed under set complement, intersection, and union. Many studies on generalized rough set approximations try to remove these restrictions. Wolski (2005) examines FCA and RST against the background of the theory of finite approximations of continuous topological spaces. The author defines operators of FCA and RST by means of the specialization order on elements of a topological space X which induces a finite approximation of X. Typically, two subsystems are used; one for lower approximation and one for upper approximation. Xu et al. (2008a) propose a generalized definition of rough set approximations, based on a subsystem of subsets of a universe. The subsystem is not assumed to be closed under set complement, union, and intersection. The lower and upper approximation is no longer one set but composed of several sets. As special cases, approximations in FCA and knowledge spaces are examined. Ganter et al. (2008b) use a generalization of the indiscernibility relation which is considered as a quasi-order of equivalence given by $g \leq h \Leftrightarrow g' \subseteq h'$ for $g, h \in G$. This generalization allows for defining version spaces and hypotheses in terms of RST and can support a feature selection process. Ganter (2008a) generalized the classical rough set approach by replacing lower and upper approximations with arbitrary kernel and closure operators respectively. Lattices of rough set abstractions were described as P-products. Meschke et al. (2009) further investigated the role of robust elements, the possible existence of suitable negation operators and the structure of corresponding lattices. Meschke (2010) builds further on this work by restricting the view for large contexts to a subcontext without losing implicational knowledge about the selected objects and attributes. Ganter and Meschke (2009, 2011) use a slightly different approach based on FCA to mine the Infobright dataset containing so-called rough tables where objects are combined together in data packs (Infobright, 2012). Their approach makes FCA useful for analyzing extremely large data.

6.3. *FCA with fuzzy attributes and rough set theory*

Pawlak's rough set model can be generalized to a fuzzy environment to deal with quantitative data. Yao (1997) introduced to this extent the notions of rough fuzzy sets and fuzzy rough sets. Since then, several papers have presented theoretical extents to this original model and applications to real data. In this section, we summarize the 15% of papers on combining FCA with fuzzy attributes with RST. One of the most thorough applications of the combination of FCA with fuzzy attributes and rough set theory was presented by Formica (2012). She used this combination to perform semantic web search. In her model, web queries are expressed by sets of attributes in the given formal context. Given a query, in case there are no formal concepts having as intent the required set of attributes, and then, the goal is to find the formal concepts of the fuzzy concept lattice whose intents better approximate the set of attributes specified by the query and therefore whose extents are closer to the expected answer. Within the various approximations obtained from the lattice, the user can additionally select the preferred one based on the

grades of membership of specific objects with specific attributes. Shao, Liu, and Zhang (2007) also study rough set approximations within FCA in a fuzzy environment and present approaches to attribute reduction and rule acquisition. Lai and Zhang (2009) present a comparative study of concept lattices of fuzzy contexts based on FCA and rough set theory. Yao and Mi (2009) study several approaches for constructing fuzzy concept lattices based on generalized fuzzy rough approximation operators. The authors then propose three kinds of fuzzy Galois connections and three kinds of lattices can be produced for which the properties are analogous to those of the classical concept lattices.

6.4. Attribute reduction

In this section, we discuss the 24% of papers on attribute reduction of concept lattices using techniques from RST. Liu et al. (2007) propose two reduction methods based on rough set theory for concept lattices. The authors apply their multistep attribute reduction method to the reduction of redundant premises of the multiple rules used in the job shop-scheduling problem. Li (2009) focuses on attribute reduction of formal concepts via covering rough set theory. Wang and Zhang (2006) discuss some of the basic relationships between the extent of concepts and the equivalence class in rough set theory. The authors also study the relation between the reductions of formal concept lattices and attribute reduction in rough set theory. Shao and Zhang (2005) introduce a pair of rough set approximations in FCA, based on both lattice-theoretic and set-theoretic operators. Algorithms for attribute and object reduction in concept lattices are also presented. Tadrat, Boonjing, and Pattaraintakorn (2007) first applies rough set theory to obtain minimally sufficient cases in the case base of a case-based reasoning system. The author then applies FCA to reveal interesting attribute dependencies by exploring the resulting concept lattices.

7. Monotone concept and AFS algebra

Deogun and Saquer (2004) discussed some of the limitations of Wille's formal concept and proposed the monotone concept. In Wille's definition of concepts, only one set is allowed as extent (intent). For many applications, it would be useful to allow intents to be defined as a disjunctive expression. The monotone concept is a generalization of Wille's notion of concept where disjunctions are allowed in the intent and set unions are allowed in the extent. This generalization allows an information retrieval query containing disjunctions to be understood as the intent of a monotone concept whose answer is the extent of that concept. By using rough set theory, Saquer et al. (2003) provided a method to find monotone concepts whose intents are close to the query and showed how to find monotone concepts whose extents approximate any given set of objects.

AFS algebra was proposed by Liu (1998) and is a new approach for the semantic interpretation of fuzzy information. In Wang and Liu (2008), the AFS formal concept is proposed, which extends the Galois connection of a regular context to the connection between two AFS algebra systems. In an information retrieval system, the logic relationships between queries are usually "and" and "or." AFS formal concepts can be used to represent queries with complex logic operations. When using an information retrieval system, we often find that not all queries are exactly contained in the database, but some items close to those are enough to satisfy the user's need. The authors study how to approximate a complex attribute by AFS formal concepts such that the intents of the lower and upper approximating concepts are close to the complex attribute's underlying

semantics. A complex attribute is compounded by some elementary attributes under logic operations "and" and "or". Zhang and Liu (2006) explore the relationships between concept lattices and AFS algebra. The authors analyze concept lattices from the point of AFS algebra. Liu and Zhang (2006) investigate the relationships between concept lattices and AFS algebra. The authors show how a concept lattice can be obtained from a given AFS algebra.

8. Conclusions

Since its introduction in 1982 as a mathematical technique, FCA became a well-known instrument in computer science. Over 1072 papers have been published over the past 9 years on FCA and many of them contained case studies showing the method's usefulness in real-life practice. This paper showcased the possibilities of FCA as a meta technique for categorizing the literature on concept analysis. The intuitive visual interface of the concept lattices allowed for an in-depth exploration of the main topics in FCA research. In particular, its combination with text mining methods resulted in a powerful synergy of automated text analysis and human control over the discovery process.

One of the most notorious research topics covering 23% of the FCA papers is KDD. FCA has been used effectively in many domains for gaining actionable intelligence from large amounts of information. Information retrieval is another important domain covering 13% of the papers. FCA was found to be an interesting instrument for representation of and navigation in large document collections and multiple IR systems resulted from this research. FCA was also used frequently (13% of papers), among others in the context of semantic web, for ontology engineering and merging. Finally, 9% of the papers devoted attention to improve FCA's applicability to larger data repositories.

In 18% of the papers, traditional concept lattices were extended to deal with uncertain and incomplete data. In particular, combining FCA with fuzzy and rough set theory received considerable attention in the literature. In this paper, we tried to summarize the literature of fuzzy and rough FCA. We identified multiple branches in the development of both theories since there advent in the early nineties. FCA with fuzzy attributes researchers were mainly interested in developing methods to control or reduce the number of extracted concepts, defining more general theories such as the multi-adjoint and generalized concept lattices and applying FCA with fuzzy attributes in domains such as IR, KDD and ontology engineering. Rough FCA researchers mainly focused on different ways for combining FCA with RST, on attribute reduction of concept lattices and combining FCA with fuzzy attributes with RST. The applications of the rough FCA theory were limited to the ontology engineering field. In the future, we will host the references and links to the articles on a public interface and hope that this compendium may serve to guide both practitioners and researchers to new and improved avenues for FCA.

Acknowledgments

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